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LETTER TO THE EDITOR

Population statistics and the counting process

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Abstract. We investigate the outcome of different monitoring schemes applied to small classical populations. Explicit calculation for a solvable model establishes the relationship between the statistical properties of the population and those of the counting processes. Parallels are drawn with operator ordering in the second quantised formulation of photodetection.

The detection of a quantised electromagnetic field is analogous to the monitoring of a classical population of individuals. The statistical properties of such a population are normally ascertained by analysing a representative (random) sample of individuals using a simple counting procedure which leaves the overall population unaffected. This is similar to the so-called quantum non-demolition measurement (Caves 1983). However, other monitoring schemes can be devised which are more closely related to detection by quantum annihilation (Glauber 1965). In this note the outcomes of two such monitoring schemes are investigated. Explicit calculations for an exactly solvable model show that the statistical properties of the counting process do not truly reflect those of the population being monitored and that there is significant loss of information in the case of small populations. The results provide further insight into the relationship between photo-electron counting statistics and the quantised nature of optical fields.

The population model and monitoring schemes to be investigated are as follows. A localised population of individuals is governed by a simple birth–death–immigration process (Bartlett 1966). The birth rate λN and death rate μN are dependent on the population N whilst the immigration rate ν is not. The death rate per individual μ is the sum of an internal process μ_i and a population dependent emigration process specified by the rate ε. In monitoring scheme 1, a counter located within the population absorbs (kills off) individuals at a (death) rate μN whilst registering the number absorbed in time intervals of duration t which may or may not be contiguous. It is clear at the outset that this scheme will change the population process and not provide a true measure of the unperturbed statistics. It is also evident that different results will be obtained according to whether the counter is switched off or not between intervals during which absorption events are registered. For simplicity it will be assumed in the following that the counter is in fact open (absorbing) all the time. In monitoring scheme 2, the counter is located outside the population and absorbs only a fraction ζ of the emigrating individuals, again registering the number absorbed during time intervals of duration t. In this case the population process is not affected and whether or not the counter actually absorbs individuals is of no consequence in the calculations. Monitoring scheme 2 is clearly a remote, non-perturbative method;

nevertheless we shall find that the counting statistics do not provide a true measure of the population statistics. This is because the counting statistics relate to a *flux* of individuals whilst the population statistics relate to the number of individuals present at a given instant of time.

The birth–death–immigration process has been investigated by many authors; however, in order to facilitate the incorporation of monitoring schemes it is useful to derive the rate equation for the process from first principles. Let $P(N;t)$ be the probability of finding N individuals in the population at time t. Then, assuming the population can only change through the gain or loss of single individuals during time intervals of sufficiently short duration Δt,

$$\begin{aligned} P(N;t+\Delta t) &= \mu(N+1)P(N+1;t)\Delta t + \lambda(N-1)P(N-1;t)\Delta t \\ &\quad + \nu P(N-1;t)\Delta t + [1-(\lambda+\mu)N\Delta t - \nu\Delta t]P(N;t). \end{aligned} \tag{1}$$

This equation expresses the probability of finding N individuals at time $t+\Delta t$ as the sum of the probabilities of a death occurring during the interval Δt in a population of $N+1$ individuals, a birth occurring or an 'immigrant' arriving during Δt in a population of $N-1$ individuals and the probability of having no change in a population of N individuals present at time t. Subtracting $P(N;t)$ from both sides, dividing by Δt and taking the limit $\Delta t \to 0$ yields the rate equation

$$\mathrm{d}P_N/\mathrm{d}t = \mu(N+1)P_{N+1} - [(\lambda+\mu)N+\nu]P_N + [\lambda(N-1)+\nu]P_{N-1} \tag{2}$$

where $P_N \equiv P(N;t)$. This equation is most easily solved by using the generating function

$$Q(z,t) = \sum_{N=0}^{\infty} (1-z)^N P_N \tag{3}$$

which satisfies the partial differential equation

$$\partial Q/\partial t = z[-\mu + \lambda(1-z)]\,\partial Q/\partial z - \nu z Q. \tag{4}$$

Now consider monitoring scheme 1. Let $P^{(1)}(n,N;t)$ be the joint probability of finding N individuals in the population at time t and of registering n 'absorptions' during the sample or integration interval $[0,t]$. Then, assuming that the probability of registering a count during a short time Δt is proportional to $\eta N\Delta t$,

$$\begin{aligned} &P^{(1)}(n,N;t+\Delta t) \\ &\quad = \mu(N+1)P^{(1)}(n,N+1;t)\Delta t + \lambda(N-1)P^{(1)}(n,N-1;t)\Delta t \\ &\qquad + \nu P^{(1)}(n,N-1;t)\Delta t + \eta(N+1)P^{(1)}(n-1,N+1;t)\Delta t \\ &\qquad + [1-(\mu+\eta+\lambda)N\Delta t - \nu\Delta t]P(n,N;t) \end{aligned} \tag{5}$$

where the absorption of an individual by the counter is accompanied by a corresponding loss from the population. A rate equation for $P^{(1)}$ can be obtained as before, but for brevity we shall set down here only the equation satisfied by the generating function

$$Q^{(1)}(s,z;t) = \sum_{n,N=0}^{\infty} (1-s)^n (1-z)^N P^{(1)}(n,N;t). \tag{6}$$

This may be written in the form

$$\partial Q^{(1)}/\partial t = z[-(\mu+\eta)+\lambda(1-z)]\,\partial Q^{(1)}/\partial z - \nu z Q^{(1)} + \eta s\,\partial Q^{(1)}/\partial z. \tag{7}$$

In monitoring scheme 2 we shall assume that the probability of registering a count is proportional to $\varepsilon\zeta\Delta t$ for sufficiently short intervals Δt. Thus if $P^{(2)}(n, n_e, N; t)$ is the joint probability of finding N individuals in the population at time t, n_e emigrants during the interval $[0, t]$ and of registering n of these emigrations, then

$$\begin{aligned}P^{(2)}(n, n_e, N; t+\Delta t) &= \mu_i(N+1)P^{(2)}(n, n_e, N+1; t)\Delta t + \lambda(N-1)P^{(2)}(n, n_e, N-1; t)\Delta t \\ &\quad + \nu P^{(2)}(n, n_e, N-1; t)\Delta t + \varepsilon(1-\zeta)(N+1)P^{(2)}(n, n_e-1, N+1; t)\Delta t \\ &\quad + \varepsilon\zeta(N+1)P^{(2)}(n-1, n_e-1, N+1; t)\Delta t \\ &\quad + [1-(\mu+\lambda)N\Delta t - \nu\Delta t]P(n, n_e, N; t). \end{aligned} \tag{8}$$

This equation can be summed over the number of emigrations, which are not of interest, and converted into a rate equation as before. Again for brevity we quote only the equation satisfied by the generating function defined as in equation (6):

$$\partial Q^{(2)}/\partial t = z[-\mu+\lambda(1-z)]\,\partial Q^{(2)}/\partial z - \nu z Q^{(2)} + \varepsilon\zeta s\,\partial Q^{(2)}/\partial z. \tag{9}$$

The general solution of equation (4) is well known. Provided that $\mu > \lambda$ the population reaches equilibrium with a negative binomial distribution

$$P_N = \binom{N-\alpha+1}{N} \frac{(\bar{N}/\alpha)^N}{(1+\bar{N}/\alpha)^{N+\alpha}} \tag{10}$$

and bilinear moment

$$\langle N(0)N(\tau)\rangle/\bar{N}^2 = 1 + (\alpha^{-1}+\bar{N}^{-1})\exp[-(\mu-\lambda)|\tau|] \tag{11}$$

where $\bar{N} = \nu/(\mu-\lambda)$ and $\alpha = \nu/\lambda$.

Recently solutions of equations of the form (7) and (9) valid for arbitrary integration time have also appeared in the literature (Jakeman 1980, Shepherd 1981). Unless this time interval is short compared to $(\mu-\lambda)^{-1}$, however, the monitoring process will average over fluctuations in the population and provide a poor measure of its statistical behaviour. Assuming then that the sample time $t \ll (\mu-\lambda)^{-1}$ we find from previous results that in equilibrium

$$P_n^{(i)} = \binom{n-\alpha+1}{n} \frac{(\bar{n}^{(i)}/\alpha)^n}{(1+\bar{n}^{(i)}/\alpha)^{n+\alpha}} \tag{12}$$

and

$$\langle n^{(i)}(0)n^{(i)}(\tau)\rangle/(\bar{n}^{(i)})^2 = 1 + \alpha^{-1}\exp(-\Gamma^{(i)}|\tau|) \tag{13}$$

where

$$\bar{n}^{(1)} = \eta\bar{N}t/(1+\eta\bar{N}/\nu), \qquad \Gamma^{(1)} = \mu+\eta-\lambda,$$
$$\bar{n}^{(2)} = \varepsilon\zeta\bar{N}t, \qquad \Gamma^{(2)} = \mu-\lambda.$$

The single interval statistics of the counting process (12) are evidently identical to those of the population process (10) apart from a simple scaling of the mean. The scaling is not linear in the case of monitoring scheme 1 because of its perturbing effect. This also results in a reduced relaxation time by comparison with the original population process (cf equations (11) and (13)). Monitoring scheme 2 is more 'satisfactory' in these respects, but in common with scheme 1 it does result in a bilinear moment which is structurally different from that of the population process: there is no term in equation (13) corresponding to the $\bar{N}^{-1}$ term on the right-hand side of equation (11). This is particularly significant in the special case $\lambda = 0$ ($\alpha \to \infty$). Equations (2) and (4) then describe a 'Poisson' process with

$$P_N = (\bar{N}^N / N!) \exp(-\bar{N}) \tag{14}$$

and

$$\langle N(0)N(\tau)\rangle / \bar{N}^2 = 1 + \bar{N}^{-1} \exp(-\mu|\tau|). \tag{15}$$

The population is Poisson distributed and exhibits fluctuations on a time scale μ^{-1} though the fluctuations will be of small amplitude if the average number of individuals is large. On the other hand, the counting process has statistical properties

$$P_n = (\bar{n}^n / n!) \exp(-\bar{n}), \tag{16}$$

$$\langle n(0)n(\tau)\rangle / \bar{n}^2 = 1, \tag{17}$$

and therefore exhibits no fluctuations!

Two special cases of the population process, equation (2), can be derived as asymptotic limits of the Scully–Lamb laser theory (Scully and Lamb 1967). The Poisson process $\lambda = 0$ considered above corresponds to a single mode laser operating well above threshold, whilst the 'Thermal' or 'Geometric' process $\nu = \lambda$ (where λ represents both stimulated and spontaneous emission rates) corresponds to a laser operating well below threshold. It is well known that in neither case do terms similar to the $\bar{N}^{-1}$ term on the right-hand side of equation (11) actually appear in theoretical expressions for the correlation functions measured by photon counting techniques. This is a direct consequence of the operator ordering used in the definition of these quantities. For example, apart from a constant of proportionality, the correlation function of photocounts for a single mode optical field is defined by the normally ordered expression (Glauber 1965):

$$G^{(2)}(\tau) = \mathrm{Tr}\, \hat{\rho}\hat{a}^+(0)\hat{a}^+(\tau)\hat{a}(\tau)\hat{a}(0), \tag{18}$$

where $\hat{a}(\tau)$ is a field mode operator in the Heisenberg picture and $\hat{\rho}$ the density matrix. Equation (18) is more transparent when expressed in the number representation (Hildred 1980):

$$G^{(2)}(\tau) = \sum_{N,M} MNP_M P(N|M-1;\tau) \tag{19}$$

where

$$P_M = \langle M|\hat{\rho}|M\rangle$$

and

$$P(N|M-1,\tau) = \langle N|e^{i\hat{L}\tau}|M-1\rangle\langle M-1|N\rangle \tag{20}$$

is the conditional probability of finding the field in state $|N\rangle$ at time $t = \tau$ given that it was in a pure state $|M-1\rangle$ at $t = 0$ ($e^{i\hat{L}\tau}$ is the time development super-operator of the field). The probability distributions appearing in (19) are analogous to those characterising the population statistics described earlier and are frequently obtained as solutions of equations similar to equation (2) as we have already mentioned. It is therefore interesting to compare (19) with the analogous expression used to define the bilinear moment of the population fluctuations and used for example to derive results (11)

$$\langle N(0)N(\tau)\rangle/\bar{N}^2 = \sum_{N,M} MNP_M P(N|M;\tau)/\bar{N}^2. \tag{21}$$

Evidently the conditional distributions on the right-hand side of equations (19) and (21) are slightly different. In fact it is not difficult to check using the known solutions of equation (2) that if formula (19) had been used to derive the bilinear moment rather than formula (21) then the $\bar{N}^{-1}$ term would not have appeared in result (11). This would have then been structurally identical to equation (13) and completely indistinguishable if monitoring scheme 2 had been adopted. Thus it appears that defining the correlation function (18) in terms of normally ordered operators may be analogous to measurement using monitoring scheme 2 discussed above. This hypothesis can be explored further by considering the definition of the bilinear moment of the counting distribution

$$\langle n(0)n(\tau)\rangle = \sum_{n,m} nmp(n, m; \tau, T), \tag{22}$$

where $p(n, m; \tau, T)$ is the joint probability of registering a count n in the sample interval T at time zero and in the sample interval at time $\tau(>T)$. This can be expressed in terms of a sum over conditional probabilities and the equilibrium population distribution P_M (Shepherd 1981):

$$p(n, m; \tau, T) = \sum_{\substack{MM'M''\\ Nm'}} P(n, M'|M; T)P(m', N|M'; \tau - T)P(m, M''|N; T)P_M, \tag{23}$$

where $P(n, N|M; T)$ is the joint probability of registering n counts in the interval $[t, t+T]$ and N individuals present in the population at time $t+T$ conditional on there being M at time t. Note that this quantity corresponds to the solution of equations (7) or (9), which are conditional on the boundary values at $t=0$, although this fact was not included in the notation used earlier for reasons of clarity. Equation (23) can be simplified for short integration times (for example $T \ll (\mu-\lambda)^{-1}$ in the above example) by observing that the terms $n, m = 0$ do not contribute to the sum (22) whilst for sufficiently small T (scheme 2)

$$nP(n, N|M; T) = \varepsilon\zeta MT\delta_{n,1}\delta_{M-1,N} + O(T^2)$$

and (24)

$$P(m', N|M'; \tau - T) = P(m', N|M'; \tau) + O(T).$$

Performing the sums over $M'M''m'$ and n and m in equation (22) we find that

$$\langle n(0)n(\tau)\rangle\bar{n}^2 = \sum_{N,M} NMP_M P(N|M-1;\tau)/\bar{N}^2 \tag{25}$$

which is structurally identical to equation (19) apart from a normalisation factor.

Thus the normal ordering of operators in the theory of photo detection is closely analogous to the adoption of flux monitoring schemes in classical population statistics. In this letter we have confined ourselves to consideration of two-fold statistics and it would be interesting to establish similar analogies in the case of higher order joint statistical properties and explore the relationship between these analogies and the Markovian assumption inherent in the calculations we have described. It is obviously possible to extend the above approach to systems in which, for example, internal population processes involve only transitions of two individuals in very short time intervals (eg two-photon transitions: see for example Loudon 1983); in that case the detection mechanisms described here would necessarily introduce single-individual events (one-photon transitions, or quantum damping, Louisell 1973) into the system, destroying the conservation of parity in population number N. Alternatively, analogous treatments can be considered in relation to the detection or counting process: two-photon detectors have already received some attention (Jaiswal and Agarwal 1969) while detectors operating through stimulated emission have been shown to possess coherence functions associated with anti-normal ordering of field operators (Mandel 1966, Perina 1971). A population-statistical analysis of these and other detection mechanisms would be valuable in helping to assess the relative merits of different systems in relation to detector noise.

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